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SOME THEOREMS ON DIOPHANTINE INEQUALITIES

par

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1. Introduction

1. Let m and n denote positive integers. Let Q denote the m-dimensional parallelepiped

(1) $$a_\mu \leqq x_\mu < b_\mu \quad (\mu = 1,2,\ldots,m),$$

where $a_1,\ldots,a_m$, $b_1,\ldots,b_m$ are integers and let P denote the n-dimensional parallelepiped

(2) $$\alpha_\nu \underset{(=)}{<} z_\nu \underset{(=)}{<} \beta_\nu \quad (\nu = 1,2,\ldots,n)$$

where $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n$ are real numbers satisfying

(3) $$\alpha_\nu < \beta_\nu \leqq \alpha_\nu + 1 \quad (\nu = 1,2,\ldots,n);$$

whereas the brackets mean that an arbitrary one of the 2^n different ways in which the signs "=" can be placed has been fixed.

Further let $f_1(x),\ldots,f_n(x)$ denote n real functions which have been defined for all lattice points $(x) = (x_1,\ldots,x_m)$ of Q.

This paper deals with the solutions $(x) \in Q$ of the simultaneous diophantine inequalities

(4) $$\alpha_\nu \underset{(=)}{<} f_\nu(x) \underset{(=)}{<} \beta_\nu \quad (\mathrm{mod}\ 1) \quad (\nu = 1,2,\ldots,n),$$

i.e. with the lattice points $(x) \in Q$ to which a lattice point $(y_1,\ldots,y_n)$ corresponds such that

(5) $$\alpha_\nu \underset{(=)}{<} f_\nu(x) - y_\nu \underset{(=)}{<} \beta_\nu \quad (\nu = 1,2,\ldots,n);$$

its purpose is to deduce an estimate for the number $N_P(Q)$ of those solutions.

In view of Weyl's well known definition of uniform distribution modulo 1, we are especially interested in the expression

(6) $$R_P(Q) = N_P(Q) - N(Q) \cdot \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu),$$

where $N(Q)$ denotes the number of lattice points $(x) \in Q$. The number $R_P(Q)$ is called the remainder in the uniform distribution of the system $(f_1, \ldots, f_n)$ (with respect to P) and the upper bound of

$$\frac{|R_P(Q)|}{N(Q)},$$

if P runs through the set of all possible parallelepipeds (2) which satisfy (3) is called the discrepancy in the uniform distribution of the system $(f_1, \ldots, f_n)$. The discrepancy will be denoted here by $D(Q)$ [1].

If in stead of one parallelepiped Q, one considers a sequence S of such parallelepipeds:

$Q_1, Q_2, \ldots$

and if to each $Q \in S$ a system $n, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ satisfying (2) and a system $f_1, \ldots, f_n$ correspond, the problem arises to study the order of magnitude of $|R_P(Q)|$, as Q runs through S. It is to be remarked that the couple of numbers m, n need not be the same for two different parallelepipeds $Q \in S$.

In the case that n is a fixed integer, independent of Q, the system $(f_1, \ldots, f_n)$ is called uniformly distributed modulo 1, if for each fixed parallelepiped P, we have

$$\frac{R_p(Q)}{N(Q)} \to 0, \text{ as } Q \text{ runs through } S.$$

It follows from some considerations of Weyl[2], that this happens, if and only if

$D(Q) \to 0$, as Q runs through S.

2. Some twenty-five years ago, J.G. van der Corput proved the following theorem 1^*, which he applied to several problems in the theory of diophantine inequalities. Neither the proof of theorem 1^*, nor the applications were published by him then, as it was his intention to publish them in the third part of a long memoir on diophantine inequalities, which was to appear in the Acta Mathematica. The first part of this memoir [3] contains an exhaustive study of Weyl 's criterion in the theory of uniform distribution modulo 1 and its applications . Whereas the theorems of that part are "qualitative", it was van der Corput's intention that the theorems of the third part ("Estimations"), would bear a quantitative character: Theorem 1^* can be considered as a more dimensional and quantitative form of the named criterion. The first part and the first half of the second part (dedicated to the theory of "rhythmic functions") of the named memoir appeared in the Acta Math. 56 and 59[4], but the publication of the other parts was delayed

by several circumstances. Nevertheless Theorem 1* has been quoted and applied several times by other authors[5].

Theorem 1*. Let S denote a sequence of parallelepipeda Q defined by (1), where a_μ and b_μ are integers. To each Q may correspond an integer n, further 2n numbers α_ν, β_ν, satisfying (3) and n real functions $f_\nu(x)$, defined for each lattice point $(x) = (x_1, \dots, x_n) \in Q$. For each $Q \in S$ we put

(7) $$T(Q;c) = \sum_{(h)}' \left| \frac{1}{N(Q)} \sum_{(x)\in Q} e^{2\pi i (h_1 f_1(x) + \cdots + h_n f_n(x))} \right| ,$$

where $\sum\limits_{(h)}'$ is to be extended over all lattice points

(8) $$(h) = (h_1, \dots, h_n) \neq (0, \dots, 0)$$

which satisfy

(9) $$|h_\nu| \leq \frac{cn}{\beta_\nu - \alpha_\nu} \log \frac{2n}{\beta_\nu - \alpha_\nu} \quad (\nu = 1, 2, \dots, n).$$

Assume that for each fixed value of $c > 0$

(10) $T(Q;c) \to 0$, if Q runs through S.

Then we have

(11) $$\frac{N_p(Q)}{\prod\limits_{\nu=1}^{n} (\beta_\nu - \alpha_\nu)\, N(Q)} \longrightarrow 1 \text{ , if Q runs through S}$$

if N(Q) denotes the number of lattice points $(x) \in Q$ and if $N_p(Q)$ denotes the number of solutions of the diophantine system

(12) $$\alpha_\nu \underset{(=)}{\leq} f_\nu(x) \underset{(=)}{\leq} \beta_\nu \quad (\text{mod } 1) \quad (\nu = 1, 2, \dots, n)$$

3. In many applications of theorem 1*, the logarithm in the right-hand member of (9) is annoying. Therefore about 1926 van der Corput posed the question to replace (9) by a weaker condition like

$$|h_\nu| \leq \frac{cn}{\beta_\nu - \alpha_\nu} \quad \text{or} \quad |h_\nu| \leq \frac{cn \log en}{\beta_\nu - \alpha_\nu} \qquad (\nu = 1, 2, \ldots, n)$$

Both of us made several attempts, which did not succeed. About fifteen years ago, we posed the question in a somewhat other direction and tried to replace theorem 1* by another theorem which would enable us to eliminate the named logarithm in all important applications. Such a theorem we proved about 1935, but it was not before 1939 that we decided to publish our proof (apart from van der Corput's paper in the Acta Mathematica) in the Compositio Mathematica. The correction of the proofsheets was finished in 1940, but then by war and occupation, the Compositio Mathematica ceased to appear. After the deliberation the printing proved to have gone lost. Our paper contained a theorem, which not only permitted to eliminate the logarithm in the wanted cases, but also gave an estimate for the remainder $R_P(Q)$ and the discrepancy $D(Q)$:

Theorem 2*. Let Q denote the parallelepiped (1), where a_μ and b_μ are integers $(\mu = 1, 2, \ldots, m)$. Let $\alpha_\nu, \beta_\nu, \lambda_\nu$ denote 3n real numbers, such that

$$\alpha_\nu < \beta_\nu \leqq \alpha_\nu + 1 \; ; \; \lambda_\nu \geqq 1 \qquad (\nu = 1,2,\dots,n)$$

and let the n real functions $f_\nu(x)$ ($\nu = 1,2,\dots,n$) be defined for all lattice points $(x) \in Q$. Let $N(Q)$ denote the number of lattice points $(x) \in Q$ and $N^*(Q)$ the number of those among them, for which the simultaneous inequalities

$$\alpha_\nu \underset{(=)}{\leq} f_\nu(x) \underset{(=)}{\leq} \beta_\nu \pmod 1 \qquad (\nu = 1,2,\dots,n)$$

hold. Put for $K > 0$

$$T(Q) = T_K(Q) = \sum_{(h)}{}^{*} p_{h_1;1} \cdots p_{h_n;n} \left| \frac{1}{N(Q)} \sum_{(x)\in Q} e^{2\pi i (h_1 f_1(x) + \dots + h_n f_n(x))} \right|,$$

where $\sum\limits_{(h)}{}^{*}$ is to be extended over all lattice points $(h) = (h_1, \dots, h_n) \neq (0, \dots, 0)$ which satisfy

(13) $$|h_\nu| \leqq 2500 \lambda_\nu \log(e\lambda_\nu) \left(\log\log(e^e \lambda_\nu)\right)^2 \qquad (\nu = 1,2,\dots,n)$$

whereas has been put

(14a) $$p_{h_\nu;\nu} = \beta_\nu - \alpha_\nu + \frac{K}{\lambda_\nu} \quad , \text{ if } h_\nu = 0 ,$$

(14b) $$p_{h_\nu,\nu} = \mathrm{Min}\left(\beta_\nu - \alpha_\nu + \frac{K}{\lambda_\nu} , \frac{K}{|h_\nu|}\right) , \text{if } 1 \leqq |h_\nu| \leqq 2\lambda_\nu ,$$

(14c) $$p_{h_\nu,\nu} = \frac{K}{|h_\nu|} e^{-\frac{|h_\nu|/\lambda_\nu}{25 \log^2 |h_\nu|/\lambda_\nu}} \quad , \text{ if } |h_\nu| > 2\lambda_\nu .$$

Then there exists a numerical constant K, such that

(15) $$\left| N^*(Q) - N(Q) \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \right| \leqq$$
$$\leqq \left\{ \prod_{\nu=1}^{n} \left(\beta_\nu - \alpha_\nu + \frac{K}{\lambda_\nu}\right) - \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \right\} N(Q) + T_K(Q) N(Q)$$

Moreover we proved that it is sufficient to put

$$K = 2 + \left\{ \int_0^1 \left(e^{-e^{\frac{1}{v}}} \cdot e^{-e^{\frac{1}{1-v}}} \right) dv \right\}^{-1}$$

but it is obvious that it is of very little use to carry out calculations of K. In my "Diophantische Approximationen" (1936) [6] I quoted the special case n=1 of theorem 2* with an outline of the proof and with some applications. Our proof was based on the following idea: If $\theta(z)$ denotes the characteristic function of the segment $\alpha \underset{(=)}{\leq} z \underset{(=)}{\leq} \beta$ and if this function is extended periodically with a period 1 over the real axis, we have

$$N^*(Q) = \sum_{(x) \in Q} \theta(f(x)) .$$

Now $\theta(z)$ is approximated by two functions $\theta_1(z)$ and $\theta_2(z)$ which are infinitely often derivable and such that

$$\theta_1 \leqq \theta \leqq \theta_2 .$$

Then, expanding θ_1 and θ_2 in their Fourier-series', one can deduce the theorem. Our proof of the full theorem 2* ($m \geqq 1$, $n \geqq 1$) went on the same line and was quite complicated.

Theorem 2* also has been quoted and applied by by A. Drewes in his thesis [7].

4. In 1949, Erdös and Turán [8)] published a theorem, which is very similar to theorem 2* in the special case m = n = 1.

Theorem 3*. If the real function $f(x)$ is defined for $x = 1,2,\ldots,N$ and if

$$\left|\sum_{x=1}^{N} e^{2\pi i h f(x)}\right| \leq \psi(h) \qquad (h = 1,2,\ldots,M),$$

where M denotes an integer ≥ 1, then for all real α, β, satisfying

$$0 \leq \alpha < \beta \leq 1,$$

we have

$$\left|N^* - (\beta-\alpha)N\right| < C\left(\frac{N}{M+1} + \sum_{h=1}^{M} \frac{\psi(h)}{h}\right),$$

where N^* denotes the number of integer solutions x in $1 \leq x \leq N$ of the inequality

$$\alpha \leq f \leq \beta \pmod 1.$$

It is clear that, if one renounces from the advantage of defining the number $\rho_h = \rho_{h_1,1}$ by (14b) i.e. by Min $(\beta - \alpha + \frac{K}{\lambda}, \frac{K}{h})$, putting $\rho_h = \frac{c}{h}$, theorem 3* is sharper than the corresponding special case m = n = 1 of theorem 2*. In their proof of theorem 3*, Erdös and Turán don't use the functions Θ_1 and Θ_2, but they use "Dunham-Jackson"-means of the Fourier series for the discontinuous function $\Theta(u)$ itself.

5. Applying the "Dunham-Jackson"-means, I now prove the following general theorem 2, which con-

siders the more dimensional case $m \geq 1$, $n \geq 1$ and which obviously is an improvement of theorem 2*. I further prove, that theorem 2 contains as a special case the following theorem 1, which is an improvement of theorem 1*, as (9) has been replaced by (9a). Further I prove that theorem 2 contains as a special case the following theorem 3, which obviously is a refinement of theorem 3*.

Theorem 1. Let S denote a sequence of parallelepipeds Q defined by (1), where a_μ and b_μ are integers. To each Q may correspond an integer n, further 2n numbers α_ν, β_ν, which satisfy (3) and n real functions $f_\nu(x)$ defined for each lattice point $(x) = (x_1, \ldots, x_m) \in Q$. For each $Q \in S$ let $T(Q;c)$ be defined by (7), where $\sum'_{(h)}$ is to be extended over all lattice points (8) which satisfy

(9a) $$|h_\nu| \leq \frac{cn \log 2n}{\beta_\nu - \alpha_\nu} \qquad (\nu = 1, 2, \ldots\ldots, n).$$

Assume that for each fixed value of $c > 0$ the relation (10) holds, if Q runs through S. Then we have (11), where $N_p(Q)$ denotes the number of solutions $(x) \in Q$ of the diophantine system (12).

Theorem 2. Let Q denote an m-dimensional parallelepiped (1), where a_μ and b_μ are integers. Let $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ denote 2 n real numbers which satisfy (3) and let the n real functions $f_1(x), \ldots, f_n(x)$ be defined for all lattice points

$(x)\in Q$. Let $\lambda_1,\dots,\lambda_n$ denote n positive numbers $\geqq 1$ and let be put $M_\nu = \lambda_\nu \log\left(e \operatorname{Min}(n,\lambda_\nu)\right)$ $(\nu = 1,2,\dots,n)$,

$$(17)\qquad T(Q) = \sum_{(h)}^{*}\left|\frac{1}{N(Q)}\sum_{(x)\in Q} e^{2\pi i(h_1 f_1(x)+\cdots\cdots+h_n f_n(x))}\right|\prod_{\nu=1}^{n} p_{h_\nu,\nu},$$

where $N(Q)$ denotes the number of lattice points $(x)\in Q$, whereas $\sum\limits_{(h)}^{*}$ has to be extended over all lattice points

$$(h) = (h_1,\dots,h_n) \neq (0,\dots,0)$$

for which

$$(18)\qquad |h_\nu| \leqq M_\nu ,$$

whereas has been put

$$p_{0,\nu} = \beta_\nu - \alpha_\nu + \frac{75}{\lambda_\nu}\ ;\quad p_{h_\nu,\nu} = \operatorname{Min}\left(\beta_\nu-\alpha_\nu+\frac{75}{\lambda_\nu},\ 1-(\beta_\nu-\alpha_\nu)+\frac{75}{\lambda_\nu},\ \frac{30}{|h_\nu|}\right)$$
$$(h_\nu\neq 0,\ \nu=1,2,\dots\dots,n).$$

Then the number $N^*(Q)$ of solutions of the diophantine system

$$(19)\qquad \alpha_\nu \underset{(=)}{\leq} f_\nu \underset{(=)}{\leq} \beta_\nu \pmod 1 \qquad (\nu=1,2,\dots\dots,n)$$

satisfies the inequality

$$(20)\qquad \left|N^*(Q)-N(Q)\prod_{\nu=1}^{n}(\beta_\nu-\alpha_\nu)\right| \leq \left\{\prod_{\nu=1}^{n}\left(\beta_\nu-\alpha_\nu+\frac{75}{\lambda_\nu}\right)-\prod_{\nu=1}^{n}(\beta_\nu-\alpha_\nu)\right\}N(Q)+T(Q)N(Q).$$

Theorem 3. If the real function $f(x)$ is defined for $x = 1,2,\dots,N$ and if

$$\left|\sum_{x=1}^{N} e^{2\pi i h f(x)}\right| \leq \psi(h) \qquad (h=1,2,\dots\dots,M),$$

where M denotes an integer $\geqq 1$, then for all real

α, β, satisfying $0<\beta-\alpha\leqq 1$,

we have

$$|N^*-(\beta-\alpha)N| \leqq \frac{150N}{M} + \sum_{h=1}^{M} p_h \psi(h),$$

with

$$p_h = \operatorname{Min}\left(\beta-\alpha+\frac{150}{M},\ 1-(\beta-\alpha)+\frac{150}{M},\ \frac{30}{h}\right).$$

§ 2. Some Lemma's.

Lemma 1. Let r and λ be positive integers and put

$$(21)\qquad R = R(r,\lambda) = \int_0^1 \left(\frac{\sin \pi\lambda t}{\sin \pi t}\right)^{2r} dt .$$

Then

$$(22)\qquad R > \frac{\lambda^{2r-1}}{2\sqrt{r}} .$$

Proof. $R = 2\int_0^{\frac{1}{2}} > \frac{2\lambda^{2r-1}}{\pi}\int_0^{\frac{1}{2}} \left(\frac{\sin\pi\lambda t}{\pi\lambda t}\right)^{2r} \pi\lambda dt$

$$= \frac{2\lambda^{2r-1}}{\pi}\int_0^{\frac{\pi\lambda}{2}} \left(\frac{\sin u}{u}\right)^{2r} du > \frac{2\lambda^{2r-1}}{\pi}\int_0^1 \left(1-\frac{u^2}{6}\right)^{2r} du$$

$$> \frac{2\lambda^{2r-1}}{\pi}\int_0^{\frac{1}{\sqrt{r}}} \left(1-\frac{u^2 r}{3}\right) du = \frac{16}{9\pi}\,\frac{\lambda^{2r-1}}{\sqrt{r}} > \frac{1}{2}\,\frac{\lambda^{2r-1}}{\sqrt{r}} .$$

Lemma 2. Let M and r be positive integers, put[9)]

$$\lambda = \lambda(M,r) = \left[\frac{M}{r}\right] + 1$$

and let $R = R(r,\lambda)$ be defined by (21). Let γ denote a number satisfying $0 \leqq \gamma \leqq 1$. Then the function

$$(23)\quad \varphi(z)=\varphi(M,r,\gamma,z)=\frac{1}{R}\int_{-z}^{\gamma-z}\left(\frac{\sin\pi\lambda t}{\sin\pi t}\right)^{2r}dt$$

which is defined for all real z, satisfies the following relations

$$(24)\quad \varphi(z)=\frac{1}{R}\int_{0}^{\gamma}\left(\frac{\sin\pi\lambda(t-z)}{\sin\pi(t-z)}\right)^{2r}dt\,;$$

$$(25)\quad \varphi(z)=\frac{1}{R}\int_{z-\gamma}^{z}\left(\frac{\sin\pi\lambda t}{\sin\pi t}\right)^{2r}dt\,;$$

$$(26)\quad 0\leqq\varphi(z)\leqq 1\quad ;\quad \varphi(z+1)=\varphi(z)\,;$$

$$(27)\quad \varphi(z)=p_0+\sum_{\substack{h=-M\\ h\neq 0}}^{+M}p_h e^{2\pi i h z}\,,$$

where

$$(28)\quad p_0=\gamma\,,$$

whereas also the numbers p_h for $h\neq 0$ don't depend on z and satisfy the inequalities

$$(29)\quad |p_h|\leqq\gamma\ ;\ |p_h|\leqq 1-\gamma\ ;\ |p_h|\leqq\frac{1}{\pi|h|}\quad (h\neq 0)\,.$$

Proof. The formula (24) is trivial by the substitution $t = u - z$ in the integral (23).
In order to prove (25) we only need to remark that the integrand in (23) is an even function of

t. Further we remark that we can write by (21)

$$(30)\quad R=\int_{-z}^{1-z}\left(\frac{\sin\pi\lambda(t-z)}{\sin\pi(t-z)}\right)^{2r}dt=\int_{0}^{1}\left(\frac{\sin\pi\lambda(t-z)}{\sin\pi(t-z)}\right)^{2r}dt,$$

as the integrand is periodic with period 1.
Therefore the first half of (26) is an immediate consequence of (24). The second half of (26) follows from (24) at once as the integrand of (24) is a periodic function of z with period 1.

We now shall prove (27). It is well known, that for all positive integers λ

$$\left(\frac{\sin(\frac{\lambda}{2}\cdot 2\pi y)}{\sin(\frac{1}{2}\cdot 2\pi y)}\right)^2=\lambda+2\sum_{h=1}^{\lambda-1}(\lambda-h)\cos 2\pi hy$$

(the empty sum for $\lambda = 1$ denoting 0).

Hence the integrand of (23) is a trigonometric polynomial in the variable $2\pi y$ of an order

$$\leqq(\lambda-1)r\leqq\left[\frac{M}{r}\right]r\leqq M .$$

Integrating, we find that $\varphi(M, r, \gamma, z)$ is a polynomial of order $\leqq M$ too, say

$$(31)\quad \varphi(z)=p_0+\sum_{h=1}^{M}(a_h\cos 2\pi hz+b_h\sin 2\pi hz)=p_0+\sum_{\substack{h=-M\\ h\neq 0}}^{+M}p_h e^{2\pi ihz},$$

where

$$(32)\quad 2p_h=a_h-ib_h\ (h>0),\ p_{-h}=a_h+ib_h\ (h>0),$$

whereas a_h, b_h, p_h don't depend on z.

We now find from (31) by (24)

$$p_0=\int_0^1 \varphi(z)\,dz=\int_0^1 \frac{dz}{R}\int_0^{\gamma}\left(\frac{\sin\pi\lambda(t-z)}{\sin\pi(t-z)}\right)^{2r}dt$$

$$=\int_0^{\gamma}\frac{dt}{R}\int_0^1\left(\frac{\sin\pi\lambda(t-z)}{\sin\pi(t-z)}\right)^{2r}dt=\gamma$$

because of (30).

By (31) and (26) we further find for $h \geqq 1$

$$a_h = \int_0^1 \varphi(z)\cos 2\pi hz\,dz = (\cos 2\pi h\xi)\int_0^1 \varphi(z)\,dz$$

$$=p_0\cos 2\pi h\xi$$

$(0 \leqq \xi \leqq 1)$ and therefore $|a_h|\leqq \gamma$.

Similarly we find $|b_h| \leqq \gamma$ and thus by (32)

$$|p_h|\leqq \gamma \qquad h\neq 0 .$$

Further we conclude from (31) and (24) for $h \geqq 1$ because of (30)

$$a_h=\int_0^1 \varphi(z)\cos 2\pi hz\,dz =$$

$$=\int_0^1\frac{\cos 2\pi hz}{R}\,dz\int_0^{\gamma}\left(\frac{\sin\pi\lambda(t-z)}{\sin\pi(t-z)}\right)^{2r}dt$$

$$=\int_0^1\frac{\cos 2\pi hz}{R}\,dz\int_0^1\left(\frac{\sin\pi\lambda(t-z)}{\sin\pi(t-z)}\right)^{2r}dt-\int_0^1\frac{\cos 2\pi hz}{R}\,dz\int_{\gamma}^1\left(\frac{\sin\pi\lambda(t-z)}{\sin\pi(t-z)}\right)^{2r}dt$$

$$= \int_0^1 \cos 2\pi hz dz - (\cos 2\pi h \xi_1) \int_0^1 \frac{dz}{R} \int_\gamma^1 \left(\frac{\sin \pi\lambda(t-z)}{\sin \pi(t-z)}\right)^{2r} dt$$

$$= -(\cos 2\pi h \xi_1) \int_\gamma^1 \frac{dt}{R} \int_0^1 \left(\frac{\sin \pi\lambda(t-z)}{\sin (t-z)\pi}\right)^{2r} dt$$

$$= -(1-\gamma) \cos 2\pi h \xi_1$$

$(0 \leqq \xi_1 \leqq 1)$. Hence

$$|a_h| \leqq 1 - \gamma .$$

Similarly we find $|b_h| \leqq 1 - \gamma$ and thus by (32)

$$|p_h| \leqq 1 - \gamma \qquad (h \neq 0) .$$

Now we shall prove the last part of (29).

Using (23) we write for $h \geqq 1$

$$a_h = \int_0^1 \varphi(z) \cos 2\pi hz dz =$$

$$= - \frac{1}{2\pi h} \int_0^1 \frac{\partial \varphi}{\partial z} \sin 2\pi hz dz =$$

$$= \frac{1}{2\pi h R} \int_0^1 \left\{ \left(\frac{\sin \pi\lambda(\gamma - z)}{\sin \pi(\gamma - z)}\right)^{2r} - \left(\frac{\sin \pi\lambda z}{\sin \pi z}\right)^{2r} \right\} \sin 2\pi hz dz$$

Hence by (30)

$$|a_h| \leqq \frac{1}{\pi h} .$$

Similarly we have $|b_h| \leqq \frac{1}{\pi h}$ and thus by (32)

$$|p_h| \leqq \frac{1}{\pi |h|} \qquad (h \neq 0) . \quad \text{Q.e.d.}$$

Lemma 3. Let M and r be positive integers, put $\lambda = \lambda(M,r) = \left[\frac{M}{r}\right] + 1$ and let $R = R(r,\lambda)$ be defined by (21). Let γ denote a number with $0 < \gamma < 1$ and let $\varphi(z) = \varphi(M, r, \gamma; z)$ be defined as in Lemma 2. Then we have

$$(33)\quad \varphi(z) \geqq 1 - \frac{2}{4^r\sqrt{r}}\left\{\frac{1}{(\lambda z)^{2r-1}} + \frac{1}{(\lambda(\gamma - z))^{2r-1}}\right\}$$

on $0<z<\gamma$

$$(34)\quad \varphi(z) \leqq \frac{2}{4^r\sqrt{r}}\left\{\frac{1}{(\lambda(z-\gamma))^{2r-1}} + \frac{1}{(\lambda(1-z))^{2r-1}}\right\}$$

on $\gamma < z < 1$.

Proof. First we suppose that $0 < z < \gamma$ and we shall prove (33). By (23) and (21) we have

$$(35)\quad \varphi(z) = \frac{1}{R}\int_{-z}^{\gamma - z}\left(\frac{\sin \pi\lambda t}{\sin \pi t}\right)^{2r} dt$$

$$= \frac{1}{R}\int_{-z}^{1-z} - \frac{1}{R}\int_{\gamma-z}^{1-z}$$

$$= 1 - \frac{1}{R}\int_{\gamma - z}^{1-z}.$$

Now

$$(36a)\quad \int_{\gamma-z}^{1-z} = \int_{\gamma-z}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1-z}, \quad \text{if } \gamma - z \leqq \tfrac{1}{2} \leqq 1-z$$

(36b) $\int_{\gamma-Z}^{1-Z} < \int_{\gamma-Z}^{\frac{1}{2}}$, if $1-Z<\frac{1}{2}$;

(36c) $\int_{\gamma-Z}^{1-Z} < \int_{\frac{1}{2}}^{1-Z}$, if $\frac{1}{2}<\gamma-Z$.

We now shall distinguish two cases.

A) Assume that $\gamma-Z \leqq \frac{1}{2}$. Then by

(37) $\sin \pi t \geqq \frac{2}{\pi} \cdot \pi t \qquad (0 \leqq t \leqq \frac{1}{2})$

we get

$$\frac{1}{R}\int_{\gamma-Z}^{\frac{1}{2}}\left(\frac{\sin\pi\lambda t}{\sin\pi t}\right)^{2r}dt \leqq \left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{\gamma-Z}^{\frac{1}{2}}\left(\frac{\sin\pi\lambda t}{\pi\lambda t}\right)^{2r}\pi\lambda\,dt$$

$$=\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{\pi\lambda(\gamma-Z)}^{\frac{\pi\lambda}{2}}\left(\frac{\sin u}{u}\right)^{2r}du$$

$$<\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{\pi\lambda(\gamma-Z)}^{\infty}\frac{du}{u^{2r}}=\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\,\frac{1}{2r-1}\,\frac{1}{\{\pi\lambda(\gamma-Z)\}^{2r-1}}$$

$$\leqq \frac{2}{4^r\sqrt{r}}\,\frac{1}{\{\lambda(\gamma-Z)\}^{2r-1}} \qquad \text{by (22)}.$$

B) Assume that $\frac{1}{2} \leqq 1-Z$. Then

$$\frac{1}{R}\int_{\frac{1}{2}}^{1-Z}\left(\frac{\sin\pi\lambda t}{\sin\pi t}\right)^{2r}dt =$$

$$=\frac{1}{R}\int_{Z}^{\frac{1}{2}}\left(\frac{\sin\pi\lambda t}{\sin\pi t}\right)^{2r}dt,$$

the integrand being symmetric on $0 \leqq t \leqq 1$; hence by (37)

$$\leqq \left(\frac{\pi\lambda}{2}\right)^{2r-1} \frac{1}{2R} \int_{z}^{\frac{1}{2}} \left(\frac{\sin \pi\lambda t}{\pi\lambda t}\right)^{2r} \pi\lambda \, dt$$

$$= \left(\frac{\pi\lambda}{2}\right)^{2r-1} \frac{1}{2R} \int_{\pi\lambda z}^{\frac{\pi\lambda}{2}} \left(\frac{\sin u}{u}\right)^{2r} du < \left(\frac{\pi\lambda}{2}\right)^{2r-1} \frac{1}{2R} \int_{\pi\lambda z}^{\infty} \frac{du}{u^{2r}}$$

$$= \left(\frac{\pi\lambda}{2}\right)^{2r-1} \frac{1}{2R} \frac{1}{2r-1} \frac{1}{(\pi\lambda z)^{2r-1}} < \frac{2}{4^r \sqrt{r}} \frac{1}{(\lambda z)^{2r-1}} \quad \text{by (22).}$$

As at least one of the cases (36a), (36b), (36c) occurs, (33) follows immediately from (35) and the results of A) and B). Q.e.d.

We now shall prove (34) and therefore assume that $\gamma < z < 1$. By (25) we have

$$(38) \qquad \varphi(z) = \frac{1}{R} \int_{z-\gamma}^{z} \left(\frac{\sin \pi\lambda t}{\sin \pi t}\right)^{2r} dt .$$

Now

$$(39\text{a}) \qquad \int_{z-\gamma}^{z} = \int_{z-\gamma}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{z} , \qquad \text{if } z-\gamma \leqq \tfrac{1}{2} \leqq z ;$$

$$(39\text{b}) \qquad \int_{z-\gamma}^{z} \leqq \int_{z-\gamma}^{\frac{1}{2}} , \qquad \text{if } z < \tfrac{1}{2} ;$$

$$(39\text{c}) \qquad \int_{z-\gamma}^{z} \leqq \int_{\frac{1}{2}}^{z} , \qquad \text{if } \tfrac{1}{2} < z-\gamma .$$

We again distinguish two cases.

A) Assume that $z-\gamma \leqq \frac{1}{2}$. Then by (37)

$$\frac{1}{R}\int_{z-\delta}^{\frac{1}{2}}\left(\frac{\sin\pi\lambda t}{\sin\pi t}\right)^{2r}dt \leqq \left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{z-\delta}^{\frac{1}{2}}\left(\frac{\sin\pi\lambda t}{\pi\lambda t}\right)^{2r}\pi\lambda dt$$

$$=\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{\pi\lambda(z-\delta)}^{\frac{\pi\lambda}{2}}\left(\frac{\sin u}{u}\right)^{2r}du$$

$$<\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{\pi\lambda(z-\delta)}^{\infty}\frac{du}{u^{2r}}$$

$$=\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\,\frac{1}{2r-1}\,\frac{1}{(\pi\lambda(z-\delta))^{2r-1}}$$

$$<\frac{2}{4^r\sqrt{r}}\,\frac{1}{\{\lambda(z-\delta)\}^{2r-1}} \quad \text{by (22).}$$

B) Assume that $\frac{1}{2}\leqq z$. Then we have by (37)

$$\frac{1}{R}\int_{\frac{1}{2}}^{z}\left(\frac{\sin\pi\lambda t}{\sin\pi t}\right)^{2r}dt = \frac{1}{R}\int_{1-z}^{\frac{1}{2}}\left(\frac{\sin\pi\lambda t}{\sin\pi t}\right)^{2r}dt$$

$$\leqq\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{1-z}^{\frac{1}{2}}\left(\frac{\sin\pi\lambda t}{\pi\lambda t}\right)^{2r}\pi\lambda\, dt$$

$$=\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{\pi\lambda(1-z)}^{\frac{\pi\lambda}{2}}\left(\frac{\sin u}{u}\right)^{2r}du<\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\int_{\pi\lambda(1-z)}^{\infty}\frac{du}{u^{2r}}$$

$$=\left(\frac{\pi\lambda}{2}\right)^{2r-1}\frac{1}{2R}\,\frac{1}{2r-1}\,\frac{1}{\{\pi\lambda(1-z)\}^{2r-1}}<\frac{2}{4^r\sqrt{r}}\,\frac{1}{\{\lambda(1-z)\}^{2r-1}}.$$

As at least one of the cases (39a), (39b), (39c) occurs, (34) follows immediately from (37) and the results of A) and B). Q.e.d.

<u>Remark.</u> In the following part of this § we often shall consider an integer $n \geqq 1$ and n positive numbers $\delta_1, \delta_2, \ldots, \delta_n$, an n-tuple of positive integers $M_1, \ldots, M_n$ and an n-tuple of positive integers $r_1, \ldots, r_n$, whereas will be put $\lambda_\nu = \left[\frac{M_\nu}{r_\nu}\right] + 1$. In this case we shall put

$$(40) \qquad \varphi_\nu(z) = \varphi\,(M_\nu, r_\nu, \delta_\nu, z),$$

where φ is the function of Lemma 2.
Further we put

$$(41) \qquad \Phi\,(z_1, \ldots, z_n) = \prod_{\nu=1}^{n} \varphi_\nu(z_\nu).$$

Then we find by Lemma 2

$$(42) \quad 0 \leqq \Phi\,(z_1, \ldots, z_n) \leqq 1$$

and

$$(43) \qquad \Phi\,(z_1, \ldots, z_n) = \delta_1 \cdots \delta_n + \sum_{(h)}^{*} p'_{h_1,1} \cdots p'_{h_n,n}\, e^{2\pi i(h_1 z_1 + \ldots + h_n z_n)},$$

where $\sum\limits_{(h)}^{*}$ is to be extended over all lattice points $(h) = (h_1, \ldots, h_n) \neq (0, \ldots, 0)$ satisfying

$$(44) \qquad |h_\nu| \leqq M_\nu \quad (\nu = 1, 2, \ldots, n),$$

whereas the $p'_{h_\nu,\nu}$ satisfy the relations

(45a) $\quad p'_{h_\nu,\nu} = \gamma_\nu \quad , \text{ if } h_\nu = 0,$

(45b) $\quad |p'_{h_\nu,\nu}| \leqq \gamma_\nu, \; |p'_{h_\nu,\nu}| \leqq 1-\gamma_\nu, \; |p'_{h_\nu,\nu}| \leqq \frac{1}{\pi |h_\nu|}, \text{ if } h_\nu \neq 0.$

Finally, if $f_1(x), \ldots, f_n(x)$ denote real functions, which have been defined for all lattice points $(x) \in Q$, where Q denotes a parallelepiped (1), we shall write

$$(46) \qquad S(h) = \sum_{(x)\in Q} e^{2\pi i (h_1 f_1 + \ldots + h_n f_n)},$$

(h) denoting a lattice point.

Lemma 4. Assumptions.

1. Let $n, M_1, \ldots, M_n$ denote positive integers, put

$$\rho_\nu = \mathrm{Min}(n, M_\nu); r_\nu = \mathrm{Max}(4, {}^2\log \rho_\nu); \lambda_\nu = \left[\frac{M_\nu}{r_\nu}\right] + 1$$
$$(\nu = 1, 2, \ldots, n).$$

2. Let Q denote a parallelepiped (1), where $a_\mu < b_\mu$ $(\mu = 1, 2, \ldots, m)$ are integers, let $f_1(x), \ldots, f_n(x)$ denote n real functions, which are defined for all lattice points $(x) = (x_1, \ldots, x_m) \in Q$. The number of lattice points $(x) \in Q$ be denoted by $N(Q)$.

3. Let P denote a parallelepiped

$$\alpha_\nu \leqq z_\nu \leqq \alpha_\nu + \delta_\nu \qquad (\nu = 1,2,\ldots\ldots,n),$$

where $\alpha_1,\ldots,\alpha_n$, $\delta_1,\ldots,\delta_n$ denote $2n$ real numbers, such that

$$0 \leqq \delta_\nu \leqq 1 \qquad (\nu = 1,2,\ldots\ldots,n).$$

Let $N_p(Q)$ denote the number of solutions $(x) \in Q$ of the system

$$(47) \qquad \alpha_\nu \leq f_\nu \leq \alpha_\nu + \delta_\nu \pmod 1 \quad (\nu = 1,2,\ldots\ldots,n).$$

Then for any given number $K \geqq 1$ we have the inequality

$$(47a) \quad N_p(Q) \leqq e^{\frac{1}{4K^7}} \left\{ \prod_{\nu=1}^{n} \left(\delta_\nu + \frac{2K+1}{\lambda_\nu}\right) N(Q) + \sum_{(h)}{}^{*} q_{h_1,1} \cdots\cdots q_{h_n,n} |S(h)| \right\},$$

where $S(h)$ is defined by (46) and where $\sum\limits_{(h)}{}^{*}$ is to be extended over all lattice points

$$(h) = (h_1,\ldots\ldots,h_n) \neq (0,\ldots\ldots,0)$$

with (44), whereas has been put

$$(48a) \qquad q_{h_\nu,\nu} = \delta_\nu + \frac{2K+1}{\lambda_\nu}, \quad \text{if } h_\nu = 0,$$

$$(48b) \qquad q_{h_\nu,\nu} = \mathrm{Min}\left(\delta_\nu + \frac{2K+1}{\lambda_\nu}, 1 - \delta_\nu + \frac{2K+1}{\lambda_\nu}, \frac{2}{\pi |h_\nu|}\right), \text{ if } h_\nu \neq 0.$$

Proof. For $\nu = 1,2,\ldots,n$ we define a function $\varphi_\nu(z_\nu)$ of the real variable z_ν as follows:

A) If $\delta_\nu < 1 - \frac{2K}{\lambda_\nu}$, we put

$$\varphi_\nu(z_\nu) = \varphi\left(M_\nu, r_\nu, \delta_\nu + \frac{2K}{\lambda_\nu}, z_\nu\right),$$

where $\varphi(z)$ is the function defined in Lemma 2 and $M_\nu, r_\nu, \delta_\nu, \lambda_\nu$, K denote the numbers which have been introduced in Lemma 4.

B) If $\delta_\nu \geqq 1 - \frac{2K}{\lambda_\nu}$, we put

$$\varphi_\nu(z_\nu) = 1 \text{ for all values of } z_\nu.$$

We now remark that in both cases:

$$\varphi_\nu(z_\nu) = p'_{0,\nu} + \sum_{\substack{h_\nu=-M_\nu \\ h_\nu \neq 0}}^{M_\nu} p'_{h_\nu,\nu}\, e^{2\pi i h_\nu z_\nu}$$

with

$$(49a) \qquad 0 \leqq p'_{0,\nu} \leqq \delta_\nu + \frac{2K}{\lambda_\nu}$$

$$(49b) \qquad |p'_{h_\nu,\nu}| \leqq \mathrm{Min}\left(\delta_\nu + \frac{2K}{\lambda_\nu}, 1-\delta_\nu, \frac{1}{\pi|h_\nu|}\right) \quad (h_\nu \neq 0).$$

In fact: in the case A) this assertion follows immediately from Lemma 2; in the case B) we put

$$p'_{0,\nu} = 1, \quad p'_{h_\nu,\nu} = 0 \quad (h_\nu \neq 0);$$

then (49a) and (49b) are trivial.

We now define $\Phi = \Phi(z_1, \dots, z_n)$ by (41), hence

$$(50) \quad \Phi(z_1, \dots, z_n) = \prod_{\nu=1}^{n} p'_{0,\nu} + \sum_{(h)}^{*} p'_{h_1,1} \cdots p'_{h_n,n}\, e^{2\pi i \sum_{\nu=1}^{n} h_\nu z_\nu},$$

where $\sum_{(h)}^{*}$ is to be extended over all lattice points $(h) = (h_1,\ldots,h_n) \neq (0,\ldots,0)$ for which (44) holds.

We now consider the special parallelepiped P^*:

$$\frac{K}{\lambda_\nu} \leqq z_\nu \leqq \delta_\nu + \frac{K}{\lambda_\nu} \qquad (\nu=1,2,\ldots,n).$$

If $(z_1,\ldots,z_n)$ belongs to P^* we have for each ν for which our case A) occurs the inequality

$$\varphi_\nu(z_\nu) > 1 - \frac{4}{4^{r_\nu}\sqrt{r_\nu}} K^{-(2r_\nu-1)}$$

by Lemma 3 and the definition of $\varphi_\nu(z_\nu)$.

But in the case B) this inequality also holds because of $\varphi_\nu(z_\nu) = 1$. Hence

$$N^*_{P^*}(Q) \prod_{\nu=1}^{n}\left(1 - \frac{4}{4^{r_\nu}\sqrt{r_\nu}} K^{-2r_\nu+1}\right) \leqq \sum_{(x)\in Q} \Phi(f_1(x),\ldots,f_n(x)).$$

Now for $0 < u < \frac{1}{2}$ we have

$$(1-u)^{-1} < 1 + 2u$$

and therefore we find, (as $r_\nu \geqq 4$)

$$(51) \quad N^*_{P^*}(Q) \leqq \left(\prod_{\nu=1}^{n} \omega_\nu\right) \sum_{x\in Q} \Phi(f_1(x),\ldots,f_n(x)),$$

where

$$\omega_\nu = 1 + \frac{4}{4^{r_\nu} K^7}:$$

<u>We now distinguish two cases with respect to ν.</u>

I. Assume that $r_\nu = {}^2\log M_\nu$. Then

$$\lambda_\nu = \left[\frac{M_\nu}{r_\nu}\right] + 1 > 4 \text{ and}$$

$$\omega_\nu = 1 + \frac{4}{M_\nu^2 K^7} < 1 + \frac{1}{4K^2\lambda_\nu}$$

because of $r_\nu \geqq 4$.

Hence we have in this case afortiori

$$\omega_\nu \leqq \mathrm{Min}\left(1 + \frac{4}{M_\nu^2}, 1 + \frac{1}{4K^2\lambda_\nu}\right).$$

II) Assume that $r_\nu = \mathrm{Max}(4, {}^2\log n)$.

We distinguish two subcases.

IIa) Let $1 \leqq n \leqq 16$. Then $r_\nu = 4$ and

$$\omega_\nu = 1 + \frac{1}{64K^7} \leqq 1 + \frac{1}{4nK^7}.$$

IIb) Let $n > 16$. Then $r_\nu = {}^2\log n$ and

$$\omega_\nu = 1 + \frac{4}{n^2K^7} < 1 + \frac{1}{4nK^7}.$$

Hence in both cases I) and II) we have estimating roughly

$$\prod_{\nu=1}^{n} \omega_\nu < \left(1 + \frac{1}{4nK^7}\right)^n \prod_{\nu=1}^{n} \mathrm{Min}\left(1 + \frac{4}{M_\nu^2}, 1 + \frac{1}{4K^2\lambda_\nu}\right).$$

Therefore we have by (51)

$$N^*_{p*}(Q) \leqq e^{\frac{1}{4K^7}} \left(\prod_{\nu=1}^{n} \mathrm{Min}\left(1 + \frac{4}{M_\nu^2}, 1 + \frac{1}{4K^2\lambda_\nu}\right)\right) \sum_{x \in Q} \Phi(f_1, \ldots, f_n).$$

and therefore, because of (50)

$$N^*_{P^*}(Q) \leq e^{\frac{1}{4K^7}} \left\{ \prod_{\nu=1}^{n} \left(p'_{o,\nu} + \frac{p'_{o,\nu}}{4K^2\lambda_\nu} \right) + \right.$$

$$\left. + \sum_{(h)}^{*} |S(h)| \prod_{\nu=1}^{n} |p'_{h_\nu,\nu}| \operatorname{Min}\left(1+\frac{4}{M_\nu^2}, 1+\frac{1}{4K^2\lambda_\nu}\right) \right\},$$

hence, in view of (49a) and (49b)

$$N^*_{P^*}(Q) \leq e^{\frac{1}{4K^7}} \left\{ \prod_{\nu=1}^{n} q_{o,\nu} + \sum_{(h)}^{*} q_{h_1,1} \cdots q_{h_n,n} |S(h)| \right\},$$

where $S(h)$ is to be extended over all lattice points $(h)=(h_1,\ldots,h_n) \neq (0,\ldots,0)$ with (44), whereas $S(h)$ is defined by (46) and $q_{h_\nu,\nu}$ by (48a) and (48b).

This proves our assertion (47a) in the special case that $P = P^*$.

Now replacing the function $f_\nu(x)$ by $f^*_\nu = f_\nu - (\alpha_\nu - \frac{K}{\lambda_\nu})$, $(\nu = 1,2,\ldots,n)$, we easily see that the point $(f^*_1,\ldots,f^*_n) \pmod 1$ lies in P^*, if and only if $(f_1,\ldots,f_n) \pmod 1$ lies in the parallelepiped P, which is defined in the third assumption of our Lemma 4. Now this translation does not inflict the value of $|S(h)|$ and therefore applying our Lemma, as proved so far, with (f^*_ν) in stead of (f_ν), we find that $N_P(Q)$, denoting the number of solutions $(x) \in Q$ of (47), satisfies the inequality (47a).

For sake of simplicity we formulate without proof the following trivial

Lemma 5. If $A_1,\dots,A_n$, $B_1,\dots,B_n$ denote 2n non-negative numbers and if P_A denotes the parallelepiped

$$0 \leqq z_\nu \underset{(=)}{\leqq} A_\nu$$

and P_{A+B} the parallelepiped

$$0 \leqq z_\nu < A_\nu + B_\nu \quad (\nu = 1,2,\dots\dots,n),$$

each point $(z_1,\dots,z_n) \in P_{A+B}$ lies in exactly one of the 2^n possible parallelepipeds

$$C_\nu \underset{((=))}{\leqq} z_\nu \underset{((=))}{\leqq} D_\nu \quad (\nu = 1,2,\dots\dots,n),$$

where either

$$C_\nu = 0,\ D_\nu = A_\nu,\ \underline{\text{or}}\ C_\nu = A_\nu,\ D_\nu = A_\nu + B_\nu \quad (\nu = 1,2,\dots\dots,n)$$

and where the double brackets mean that the signs = have to be placed in a suitable way. Ranging these 2^n parallelepipeds in some order, we shall denote them by H_ξ ($\xi = 1,2,\dots,2^n$) such that $P_A = H_1$, whereas H_{2^n} denotes the parallelepiped

$$A_\nu \leqq z_\nu \leqq A_\nu + B_\nu \quad (\nu = 1,2,\dots\dots,n).$$

The sum of the volumes of all H_ξ's is $\prod_{\nu=1}^{n}(A_\nu + B_\nu)$; arithmetically:

$$\sum \delta_1\delta_2\dots\dots\delta_n = \prod_{\nu=1}^{n}(A_\nu + B_\nu),$$

where the left hand sum is to be extended over all possible products with $\delta_\nu = A_\nu$ or $\delta_\nu = B_\nu$.

Now we shall prove

Lemma 6. Let the assumptions 1 and 2 of Lemma 4 be valid. Let $\gamma_1, \dots, \gamma_n$ denote n real numbers, such that $0 \leq \gamma_\nu \leq 1$ 1 and let P denote the parallelepiped

$$\alpha_\nu < z_\nu < \alpha_\nu + \gamma_\nu \qquad (\nu = 1,2,\dots,n)$$

Then the number $N_p(Q)$ of solutions $(x) \in Q$ of the inequalities

$$\alpha_\nu < f_\nu < \alpha_\nu + \gamma_\nu \pmod 1 \quad (\nu = 1,2,\dots,n)$$

satisfies the condition

$$(52) \quad N_P(Q) \geq \gamma_1\gamma_2\cdots\gamma_n N(Q) - \left\{\prod_{\nu=1}^{n}\left(\gamma_\nu + \frac{75}{\lambda_\nu}\right) - \prod_{\nu=1}^{n}\gamma_\nu\right\} N(Q)$$

$$- \sum_{(h)}^{*} p_{h_1,1}\, p_{h_2,2} \cdots p_{h_n,n} |S(h)|,$$

where $S(h)$ is defined by (46) and where $\sum\limits_{(h)}^{*}$ is to be extended over all lattice points

$$(h) = (h_1, \dots, h_n) \neq (0, \dots, 0)$$

which satisfy (44), whereas has been put

$$(53a) \quad p_{0,\nu} = \gamma_\nu + \frac{75}{\lambda_\nu} \qquad (\nu = 1,2,\dots,n)$$

(53b) $P_{h_\nu;\nu} = \mathrm{Min}\left(\gamma_\nu + \frac{75}{\lambda_\nu}, 1-\gamma_\nu + \frac{75}{\lambda_\nu}, \frac{30}{|h_\nu|}\right) \quad (h_\nu \neq 0)$

$$(\nu=1,2,\ldots\ldots,n).$$

Proof. We first restrict ourselves to the parallelepiped P_o for which $\alpha_\nu = 0$ $(\nu=1,2,\ldots,n)$. We put (according to (40) and (41))

$$\Phi(z_1,\ldots\ldots,z_n) = \prod_{\nu=1}^{n} \varphi(M_\nu, r_\nu, \gamma_\nu, z_\nu),$$

where φ denotes the function which we have introduced in Lemma 2, whereas the numbers M_ν, r_ν, γ_ν have been defined in Lemma 6. Then we write

(54) $\sum_{(x)\in Q} \Phi(f_1(x),\ldots\ldots,f_n(x)) = \Sigma_1 + \Sigma_2 + \Sigma_3$,

where Σ_1 is to be extended over all $N_{P_o}(Q)$ solutions $(x) \in Q$ of the inequalities

(55) $\quad 0 < f_\nu < \gamma_\nu \quad (\nu=1,2,\ldots\ldots,n)$,

where Σ_2 is to be extended over all solutions $(x) \in Q$ of the inequalities

(56) $\quad 0 \leqq f_\nu < \gamma_\nu \quad (\nu=1,2,\ldots\ldots,n)$,

which do not satisfy (55), whereas Σ_3 is to be extended over all other lattice points $(x)\in Q$. Now we have, as $0 \leqq \Phi \leqq 1$, clearly

$$N_{P_o}(Q) \geqq \Sigma_1 ;$$

hence by (54)

(57) $N_{P_0}(Q) \geqq \sum_{(x)\in Q} \Phi(f_1,\ldots\ldots,f_n) - \Sigma_2 - \Sigma_3$.

Now by (43) we have

(58) $$\Phi(z_1,\ldots\ldots,z_n) = \prod_{\nu=1}^{n} \varphi(M_\nu, r_\nu, \delta_\nu, z_\nu) =$$

$$= \prod_{\nu=1}^{n} \delta_\nu + \sum_{(h)}{}^{*} p'_{h_1,1} p'_{h_2,2} \cdots\cdots p'_{h_n,n} e^{2\pi i(h_1 z_1 + \cdots\cdots + h_n z_n)},$$

where $\sum_{(h)}^{*}$ is to be extended over all lattice points

(59) $$(h) = (h_1,\ldots\ldots,h_n) \neq (0,\ldots\ldots,0),$$

which satisfy (44), whereas (45a) and (45b) hold. Hence

(60) $$\sum_{(x)\in Q} \Phi(f_1(x),\ldots\ldots,f_n(x)) \geqq \delta_1 \delta_2 \cdots\cdots \delta_n N(Q) +$$

$$- \sum_{(h)}{}^{*} p''_{h_1,1} p''_{h_2,2} \cdots\cdots p''_{h_n,n} |S(h)|,$$

where $\sum_{(h)}^{*}$ is to be extended over all lattice points (59) which satisfy (44), where has been put

(61) $$p''_{0,\nu} = \delta_\nu , \quad p''_{h_\nu,\nu} = \mathrm{Min}\left(\delta_\nu, 1-\delta_\nu, \frac{1}{\pi |h_\nu|}\right) \qquad (h_\nu \neq 0),$$

whereas $S(h)$ denotes the sum (46).

We now shall give an estimate for the sum $\sum_2$. Every point $(z_1,\ldots,z_n)$, which lies in the parallelepiped

(62) $0 \leqq z_\nu < \gamma_\nu \qquad (\nu = 1,2,\ldots,n),$

but not lying in

(63) $0 < z_\nu < \gamma_\nu \qquad (\nu = 1,2,\ldots,n)$

lies in at least one of the n parallelepipeds P':

$$0 \leqq z_\nu \leqq \delta_\nu \qquad (\nu = 1,2,\ldots,n),$$

where $\delta_\nu = 0$ for exactly one value of ν and $\delta_\nu = \gamma_\nu$ otherwise.

Applying Lemma 4 with K = 1 and $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, we find, that the number $N_{P'}(Q)$ for each such a parallelepiped P' satisfies the inequality (47a). Summing over all P' we find that the number of terms of the sum $\sum_2$ is

$$\leqq 2N(Q)\sum_{P'}\prod_{\nu=1}^{n}\left(\delta_\nu+\frac{3}{\lambda_\nu}\right)+2\sum_{(h)}{}^{*}|S(h)|\sum_{P'}q_{h_1,1}\cdot\ldots\cdot q_{h_n,n}.$$

Using the idea of lemma 5 one easily sees that this expression is

$$\leqq \frac{1}{2}N(Q)\left\{\prod_{\nu=1}^{n}\left(\gamma_\nu+\frac{15}{\lambda_\nu}\right)-\prod_{\nu=1}^{n}\left(\gamma_\nu+\frac{3}{\lambda_\nu}\right)\right\}$$
$$+\frac{1}{2}\sum_{(h)}{}^{*}|S(h)|\left\{q^{*}_{h_1,1}\cdot\ldots\cdot q^{*}_{h_n,n}-q_{h_1,1}\cdot\ldots\cdot q_{h_n,n}\right\},$$

where

$$q^*_{o,\nu} = \gamma_\nu + \frac{15}{\lambda_\nu}, \quad q^*_{h_\nu,\nu} = \mathrm{Min}\left(\gamma_\nu + \frac{15}{\lambda_\nu}, 1-\gamma_\nu + \frac{15}{\lambda_\nu}, \frac{10}{|h_\nu|}\right)$$
$$(h_\nu \neq o),$$

whereas the numbers $q_{h_\nu,\nu}$ are defined by (48a) and (48b) with $\delta_\nu = \gamma_\nu$ (ν = 1,2,...,n).
As each term in $\sum_2$ is $\geqq$ 0 and $\leqq$ 1, we thus find a fortiori

$$(64) \qquad 0 \leqq \sum\nolimits_2 \leqq \tfrac{1}{2} N(Q) \left\{ \prod_{\nu=1}^{n} \left(\gamma_\nu + \frac{75}{\lambda_\nu}\right) - \prod_{\nu=1}^{n} (\gamma_\nu) \right\}$$
$$+ \tfrac{1}{2} \sum\nolimits^* \left(p_{h_1,1} \cdots p_{h_n,n} - p''_{h_1,1} \cdots p''_{h_n,n}\right) |S(h)|,$$

where $p_{h_\nu,\nu}$ has been defined by (53a) and (53b) and $p''_{h_\nu,\nu}$ by (61).

Finally we shall deduce an estimate for $\sum_3$. Each point ($z_1, \ldots, z_n$) which lies in the unit cube $0 \leqq z_\nu < 1$ (ν = 1,2,...n), but does not ly in the parallelepiped (62), belongs to exactly one of the 2^n - 1 parallelepipeds $H_2, H_3, \ldots, H_{2^n}$, which we have defined in Lemma 5, putting there

$$A_\nu = \delta_\nu \,, \quad B_\nu = 1 - \delta_\nu \,.$$

Take a fixed such a parallelepiped H_ξ $(2 \leq \xi \leq 2^n)$, given by the inequalities

$$C_\nu^{(\xi)} \leqq z_\nu < D_\nu^{(\xi)} \qquad (\nu = 1, 2, \ldots\ldots, n)$$

and let $\mu = \mu_1, \mu_2, \ldots, \mu_s$ denote the values of the index ν for which

$$C_\nu^{(\xi)} = 0 \,, \ D_\nu^{(\xi)} = \delta_\nu \,,$$

whereas $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_{n-s}$ may denote the values of ν for which

$$C_\nu^{(\xi)} = \delta_\nu \,, D_\nu^{(\xi)} = 1 \,.$$

Then we have

$$s \geqq 0 \,, \ n - s \geqq 1 \,.$$

Now we cover H_ξ by parallelepipeds

$$H_\xi(k) = H_\xi(k_{\sigma_1}, k_{\sigma_2}, \ldots\ldots, k_{\sigma_{n-s}})$$

defined by

$$(65) \qquad 0 \leqq z_{\mu_i} < \delta_{\mu_i} \qquad (i = 1, 2, \ldots\ldots, s) \,;$$

$$(66) \qquad \delta_{\sigma_j} + \frac{2k_{\sigma_j}}{\lambda_{\sigma_j}} \leqq z_{\sigma_j} < \delta_{\sigma_j} + \frac{2(k_{\sigma_j}+1)}{\lambda_{\sigma_j}} \quad (j = 1, 2, \ldots\ldots, n-s) \,,$$

where k_{σ_j} is an integer which satisfies

(67) $$0 \leqq k_{\sigma_j} \leqq K_j ,$$

K_j denoting the smallest non-negative integer
$$\geqq \frac{\lambda_{\sigma_j}(1-\gamma_{\sigma_j})}{2} - 1 .$$

In the special case that for some j we have $\frac{2}{\lambda_{\sigma_j}} > 1-\gamma_{\sigma_j}$, it is clear that $K_j = 0$. In this case we replace (66) by

(66a) $$\gamma_{\sigma_j} \leqq z_{\sigma_j} < 1 .$$

We now shall apply Lemma 4 with

$P = H_{\xi}(k)$, where $(k_{\sigma_1}, \ldots, k_{\sigma_{n-s}})$ denotes a fixed lattice point satisfying (67).
Hence

$$\alpha_{\mu_i} = 0, \quad \delta_{\mu_i} = \gamma_{\mu_i} \quad (i = 1,2,\ldots,s),$$

$$\alpha_{\sigma_j} = \gamma_{\sigma_j} + \frac{2k_{\sigma_j}}{\lambda_{\sigma_j}} , \quad \delta_{\sigma_j} = \mathrm{Min}\left(\frac{2}{\lambda_{\sigma_j}}, 1-\gamma_{\sigma_j}\right) \quad (j=1,2,\ldots,n-s) .$$

Clearly the numbers δ_ν $(\nu = 1,2,\ldots,n)$ satisfy the relation $0 \leqq \delta_\nu \leqq 1$. They don't depend on the values of k_{σ_j} , but only on ξ . We express this dependence by writing $\delta_\nu^{(\xi)}$ in stead of δ_ν . Writing $N(Q;H_{\xi}(k))$ in stead of $N_P(Q)$, we find from Lemma 4, putting $K = 1$:

$$(68)\quad N(Q;H_\xi(k)) \leqq 2\Big\{\prod_{\nu=1}^{n}\Big(\delta_\nu^{(\xi)}+\frac{3}{\lambda_\nu}\Big)N(Q) + \sum_{(h)}{}^{*} q_{h_1,1}\cdots\cdots q_{h_n,n}|S(h)|\Big\},$$

where $\sum_{(h)}^{*}$ is to be extended over all lattice points (59), which satisfy (44), whereas the numbers $q_{h_\nu,\nu}$ and $S(h)$ has been defined in Lemma 4.

From (68) we conclude a fortiori

$$(68a)\qquad N(Q;H_\xi(k)) \leqq B_\xi^{*},$$

where has been put

$$(69)\quad B_\xi^{*} = 2\prod_{\nu=1}^{n}\Big(\eta_\nu^{(\xi)}+\frac{3}{\lambda_\nu}\Big)N(Q) + 2\sum_{(h)}{}^{*} q_{h_1,1}^{(\xi)}\cdots\cdots q_{h_n,n}^{(\xi)}|S(h)|;$$

with

$$\eta_{\mu_i}^{(\xi)} = \gamma_{\mu_i} \quad (i=1,2,\ldots\ldots,s),$$

$$\eta_{\sigma_j}^{(\xi)} = \frac{2}{\lambda_{\sigma_j}} \quad (j=1,2,\ldots\ldots,n\text{-}s),$$

$$q_{0,\mu_i}^{(\xi)} = \gamma_{\mu_i}+\frac{3}{\lambda_{\mu_i}}\;;\; q_{h_{\mu_i},\mu_i}^{(\xi)} = \mathrm{Min}\Big(\gamma_{\mu_i}+\frac{3}{\lambda_{\mu_i}}, 1-\gamma_{\mu_i}+\frac{3}{\lambda_{\mu_i}}, \frac{2}{|h_{\mu_i}|}\Big)$$
$$(h_{\mu_i}\neq 0)\quad (i=1,2,\ldots\ldots,s),$$

$$q_{0,\sigma_j}^{(\xi)} = \frac{5}{\lambda_{\sigma_j}}\;;\; q_{h_{\sigma_j},\sigma_j}^{(\xi)} = \mathrm{Min}\Big(\frac{5}{\lambda_{\sigma_j}}, \frac{2}{|h_{\sigma_j}|}\Big)$$
$$(h_{\sigma_j}\neq 0)\quad (j=1,2,\ldots\ldots,n\text{-}s).$$

It is clear that both terms of B_ξ^* only depend on ξ and not on the lattice point $(k_{\sigma_1},\dots,k_{\sigma_{n-s}})$ which defines $H_\xi(k)$. Hence B_ξ^* is the same number for all parallelepipeds $H_\xi(k)$ which cover H_ξ.
We now consider the value of

$$\Phi(z_1,\dots\dots,z_n)=\prod_{\nu=1}^{n}\varphi(M_\nu,r_\nu,\gamma_\nu,z_\nu)$$

for a point $(z_1,\dots,z_n)\in H_\xi(k)$.
We always have

$$0\leqq\varphi(M_\nu,r_\nu,\gamma_\nu,z_\nu)\leqq 1$$

and, if

$$1\leqq k_{\sigma_j}\leqq K_j-2\,,$$

we have by Lemma 3 for $\nu=\sigma_j$

$$0\leqq\varphi(M_\nu,r_\nu,\gamma_\nu,z_\nu)\leqq\frac{4}{4^{r_\nu}\sqrt{r_\nu}}\left\{\frac{1}{k_{\sigma_j}^{2r_\nu-1}}+\frac{1}{(K_j-k_{\sigma_j})^{2r_\nu-1}}\right\}$$

$$\leqq\frac{1}{128}\left\{\frac{1}{k_{\sigma_j}^{7}}+\frac{1}{(K_j-k_{\sigma_j})^{7}}\right\}$$

by $r_\nu\geqq 4$ (cf the definition of $H_\xi(k)$ by (65) and (66).
Hence because of (68a) the sum

$$\sum{}''\Phi(f_1(x),\dots\dots,f_n(x))\,,$$

where $\sum''$ is to be extended over the $N(Q,H_\xi(k))$ lattice points $(x)\in Q$, for which $(f_1,\dots,f_n)$ (mod 1) lies in $H_\xi(k)$, satisfies the inequality

$$\sum{}'' \Phi(f_1,\ldots\ldots,f_n) \leqq B^*_{\xi} \prod_{j=1}^{n-s} \Omega_j(k_{\sigma_j}),$$

where

$$\Omega_j(k_{\sigma_j}) = 1 \text{ , if } k_{\sigma_j} = 0, \text{ or if } k_{\sigma_j} = K_j - 1, \text{ or} = K_j \text{ ,}$$

$$\Omega_j(k_{\sigma_j}) = \frac{1}{128}\left\{\frac{1}{k_{\sigma_j}^7} + \frac{1}{(K_j - k_{\sigma_j})^7}\right\}, \text{ if } 1 \leqq k_{\sigma_j} \leqq K_j - 2 .$$

Therefore

$$\sum{}^{(\xi)} \Phi(f_1(x),\ldots\ldots,f_n(x)),$$

where $\sum^{(\xi)}$ has to be extended over all lattice points $(x) \in Q$, for which $(f_1,\ldots,f_n) \pmod 1$ lies in H_ξ, satisfies the inequality

$$\sum{}^{(\xi)} \Phi(f_1,\ldots\ldots,f_n) \leqq \sum_{(k)} \sum{}'' \leqq B^*_\xi \sum_{(k)} \prod_{j=1}^{n-s} \Omega_j(k_{\sigma_j}),$$

where $\sum_{(k)}$ has to be extended over all lattice points $(k) = (k_{\sigma_1},\ldots,k_{\sigma_{n-s}})$ which satisfy (67).

Hence a fortiori

$$\sum{}^{(\xi)} \Phi(f_1,\ldots\ldots,f_n) \leqq B^*_\xi \prod_{j=1}^{n-s}\left(\sum_{(k)} \Omega_j(k_{\sigma_j})\right)$$

$$\leqq B^*_\xi \prod_{j=1}^{n-s}\left(3 + \frac{1}{64}\sum_{k=1}^{\infty}\frac{1}{k^7}\right)$$

$$\leqq B^*_\xi \left\{3 + \frac{1}{64}\left(1 + \int_1^\infty \frac{du}{u^7}\right)\right\}^{n-s} < \left(\frac{7}{2}\right)^{n-s} B^*_\xi .$$

Using this result, we find from (69) and $n-s \geqq 1$

$$\sum^{(\xi)} \Phi(f_1,\ldots\ldots,f_n) \leqq 7^{n-s} \prod_{\nu=1}^{n} \left(\eta_\nu^{(\xi)} + \frac{3}{\lambda_\nu}\right) N(Q)$$

$$+ 7^{n-s} \sum_{(h)}^{*} q_{h_1,1}^{(\xi)} \cdots\cdots q_{h_n,n}^{(\xi)} |S(h)|$$

and a fortiori

$$\leqq \frac{1}{2} \prod_{\nu=1}^{n} \varrho_\nu^{(\xi)} \cdot N(Q) + \frac{1}{2} \sum_{(h)}^{*} p_{h_1,1}^{(\xi)} \cdots\cdots p_{h_n,n}^{(\xi)} |S(h)|,$$

where has been put

$$\varrho_{\mu_i}^{(\xi)} = \eta_{\mu_i}^{(\xi)} = \gamma_{\mu_i} + \frac{3}{\lambda_{\mu_i}} \quad (i = 1,2,\ldots\ldots,s),$$

$$\varrho_{\sigma_j}^{(\xi)} = 14\left(\eta_{\sigma_j}^{(\xi)} + \frac{3}{\lambda_{\sigma_j}}\right) = \frac{70}{\lambda_{\sigma_j}} \quad (j = 1,2,\ldots\ldots,n-s),$$

$$p_{0,\mu_i}^{(\xi)} = q_{0,\mu_i}^{(\xi)} = \gamma_{\mu_i} + \frac{3}{\lambda_{\mu_i}}; \; p_{h_{\mu_i},\mu_i}^{(\xi)} = q_{h_{\mu_i},\mu_i}^{(\xi)}$$

$$= \mathrm{Min}\left(\gamma_{\mu_i} + \frac{3}{\lambda_{\mu_i}}, 1-\gamma_{\mu_i} + \frac{3}{\lambda_{\mu_i}}, \frac{2}{|h_{\mu_i}|}\right) \; (h_{\mu_i} \neq 0)$$

$$p_{0,\sigma_j}^{(\xi)} = 14 q_{0,\sigma_j}^{(\xi)} = \frac{70}{\lambda_{\sigma_j}}; \; p_{h_{\sigma_j},\sigma_j}^{(\xi)} = 14\, q_{h_{\sigma_j},\sigma_j}^{(\xi)}$$

$$= \mathrm{Min}\left(\frac{70}{\lambda_{\sigma_j}}, \frac{28}{|h_{\sigma_j}|}\right) \; (h_{\sigma_j} \neq 0).$$

Consequently we find

$$\sum_3 \Phi(f_1,\dots,f_n) = \sum_{\xi=2}^{2^n} \sum^{(\xi)} \Phi(f_1,\dots,f_n)$$

$$\leq \tfrac{1}{2} N(Q) \sum_{\xi=2}^{2^n} \prod_{\nu=1}^{n} \varsigma_\nu^{(\xi)} + \tfrac{1}{2} \sum_{(h)}^{*} |S(h)| \sum_{\xi=2}^{2^n} p_{h_1,1}^{(\xi)} \cdots\cdots p_{h_n,n}^{(\xi)} .$$

In order to carry out the summation over ξ , we use Lemma 5. Putting there

$$A_\nu = \gamma_\nu + \frac{3}{\lambda_\nu} \; , \; B_\nu = \frac{70}{\lambda_\nu} \; ,$$

we find

$$\sum_{\xi=2}^{2^n} \prod_{\nu=1}^{n} \varsigma_\nu^{(\xi)} = \prod_{\nu=1}^{n} \left(\gamma_\nu + \frac{73}{\lambda_\nu}\right) - \prod_{\nu=1}^{n} \left(\gamma_\nu + \frac{3}{\lambda_\nu}\right)$$

and putting for a fixed lattice point $(h) \neq (0,..,0)$ with (44)

$$A_\nu = \gamma_\nu + \frac{3}{\lambda_\nu} \text{, if } h_\nu = 0 \; ; \; A_\nu = \operatorname{Min}\left(\gamma_\nu + \frac{3}{\lambda_\nu}, 1 - \gamma_\nu + \frac{3}{\lambda_\nu}, \frac{2}{|h_\nu|}\right)$$
$$\text{if } h_\nu \neq 0,$$

$$B_\nu = \frac{70}{\lambda_\nu} \text{, if } h_\nu = 0; \; B_\nu = \operatorname{Min}\left(\frac{70}{\lambda_\nu}, \frac{28}{|h_\nu|}\right), \text{ if } h_\nu \neq 0,$$

we find

$$\sum_{\xi=2}^{2^n} p_{h_1,1}^{(\xi)} \cdots\cdots p_{h_n,n}^{(\xi)} = \prod_{\nu=1}^{n} p_{h_\nu,\nu}^{**} - \prod_{\nu=1}^{n} p_{h_\nu,\nu}^{*} ,$$

where

$$p^{**}_{0,\nu} = \gamma_\nu + \frac{73}{\lambda_\nu}\,, \quad p^{**}_{h_\nu,\nu} = \operatorname{Min}\left(\gamma_\nu + \frac{73}{\lambda_\nu}, 1-\gamma_\nu + \frac{73}{\lambda_\nu}, \frac{30}{|h_\nu|}\right),$$

$$(h_\nu \neq 0),$$

$$p^{*}_{0,\nu} = \gamma_\nu + \frac{3}{\lambda_\nu}\,, \quad p^{*}_{h_\nu,\nu} = \operatorname{Min}\left(\gamma_\nu + \frac{3}{\lambda_\nu}, 1-\gamma_\nu + \frac{3}{\lambda_\nu}, \frac{2}{|h_\nu|}\right),$$

$$(h_\nu \neq 0).$$

Hence, we have a fortiori

$$(70) \quad 0 \leqq \sum\nolimits_3 \Phi(f_1,\dots\dots,f_n) \leqq \frac{1}{2} N(Q)\left\{\prod_{\nu=1}^{n}\left(\gamma_\nu + \frac{75}{\lambda_\nu}\right) - \prod_{\nu=1}^{n}\gamma_\nu\right\}$$

$$+ \frac{1}{2}\sum_{(h)}{}^{*} |S(h)| \left\{p_{h_1,1}\cdots\cdots p_{h_n,n} - p''_{h_1,1}\cdots\cdots p''_{h_n,n}\right\},$$

where $p_{h_\nu,\nu}$ is defined by (53a) and (53b) whereas $p''_{h_\nu,\nu}$ is defined by (61).

From (64) and (70) we deduce

$$0 \leqq \sum\nolimits_2 + \sum\nolimits_3 \leqq N(Q)\left\{\prod_{\nu=1}^{n}\left(\gamma_\nu + \frac{75}{\lambda_\nu}\right) - \prod_{\nu=1}^{n}\gamma_\nu\right\}$$

$$+ \sum_{(h)}{}^{*}\left\{p_{h_1,1}\cdots\cdots p_{h_n,n} - p''_{h_1,1}\cdots\cdots p''_{h_n,n}\right\}|S(h)|$$

and therefore it follows from (57) and (60) that

$$N_{P_0}(Q) \geqq \sum_{(x)\in Q} - \sum\nolimits_2 - \sum\nolimits_3$$

$$\geqq \gamma_1\gamma_2\dots\dots\gamma_n N(Q) - \left\{\prod_{\nu=1}^{n}\left(\gamma_\nu + \frac{75}{\lambda_\nu}\right) - \prod_{\nu=1}^{n}\gamma_\nu\right\} N(Q) - \sum_{(h)}{}^{*} p_{h_1,1}\cdots\cdots p_{h_n,n}|S(h)|,$$

which proves (52) in the special case $P = P_0$. Replacing f_ν by $f_\nu - \alpha_\nu$, we remark that this does not inflict the right hand member of (52). Hence the theorem follows at once for an arbitrary system of inequalities $\alpha_\nu < f_\nu < \beta_\nu \pmod 1$ instead of the special system (55). Q.e.d.

Lemma 7. Let the assumptions 1 and 2 of Lemma 4 be valid. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ denote n real numbers, such that $0 \leqq \gamma_\nu \leqq 1$ and let P denote the parallelepiped

(71) $$\alpha_\nu \leqq z_\nu \leqq \alpha_\nu + \gamma_\nu \qquad (\nu = 1, 2, \ldots, n).$$

Then the number $N_P(Q)$ of solutions $(x) \in Q$ of the inequalities

(72) $$\alpha_\nu \leqq f_\nu \leqq \alpha_\nu + \gamma_\nu \pmod 1 \quad (\nu = 1, 2, \ldots, n)$$

satisfies the condition

(73) $$N_P(Q) \leqq \gamma_1 \gamma_2 \cdots \gamma_n N(Q) + \left\{ \prod_{\nu=1}^{n} \left(\gamma_\nu + \frac{75}{\lambda_\nu} \right) - \prod_{\nu=1}^{n} \gamma_\nu \right\} N(Q)$$

$$+ \sum_{(h)}^{*} \rho_{h_1,1} \cdots \rho_{h_n,n} |S(h)|,$$

where $S(h)$ is defined by (46) and where $\sum\limits_{(h)}^{*}$ is to be extended over all lattice points

$$(h) = (h_1, \ldots, h_n) \neq (0, \ldots, 0),$$

which satisfy (44), whereas has been put

(74a) $$p_{0,\nu} = \gamma_\nu + \frac{75}{\lambda_\nu} \quad (\nu=1,2,\ldots\ldots,n)$$

(74b) $$p_{h_\nu,\nu} = \operatorname{Min}\left(\gamma_\nu + \frac{75}{\lambda_\nu}, 1-\gamma_\nu + \frac{75}{\lambda_\nu}, \frac{30}{|h_\nu|}\right) \quad (h_\nu \neq 0).$$

Proof. We first restrict ourselves to the parallelepiped P_0 for which $\alpha_\nu = 0$ ($\nu = 1,2,\ldots,n$). We put, according to (40) and (41)

(75) $$\Phi = \Phi(z_1,\ldots\ldots,z_n) = \prod_{\nu=1}^{n} \varphi(M_\nu, r_\nu, \gamma_\nu, z_\nu),$$

where φ denotes the function which we have introduced in Lemma 2, whereas the numbers M_ν, r_ν have been introduced by the assumptions of Lemma 4, which hold in our case.

Now let $\sum^*$ denote a sum which is to be extended over all lattice points $(x) \in Q$ for which the system $f_1, \ldots, f_n$ satisfies the inequalities

(76) $$0 \leqq f_\nu \leqq \gamma_\nu \pmod 1 \quad (\nu=1,2,\ldots\ldots,n).$$

Then the number $N_{P_0}(Q)$ of those lattice points satisfies the following relation

(77) $$\begin{cases} N_{P_0}(Q) = \sum^* 1 = \sum^* \Phi(f_1,\ldots\ldots,f_n) + \sum^* \left(1-\Phi(f_1,\ldots\ldots,f_n)\right) \\ \leqq \sum_{(x)\,\mathrm{in}\,Q} \Phi(f_1,\ldots\ldots,f_n) + \sum^* \left(1-\Phi(f_1,\ldots\ldots,f_n)\right) \end{cases}$$

because of (42). Now from (43) and (77) it immediately follows:

$$(78)\quad N_{P_0}(Q) \leqq \gamma_1\gamma_2\cdots\gamma_n N(Q) + \sum_{(h)}{}^{*} p'_{h_1,1}\cdots p'_{h_n,n}|S(h)| + \sum{}^{*}\left(1-\Phi(f_1,\ldots,f_n)\right),$$

where $S(h)$ denotes the sum which is defined by (46) and where $\sum\limits_{(h)}^{*}$ is to be extended over all lattice points $(h) = (h_1,\ldots,h_n) \neq (0,\ldots,0)$, satisfying (44), whereas has been put

$$(79)\quad p'_{0,\nu} = \gamma_\nu,\quad p'_{h_\nu,\nu} = \mathrm{Min}\left(\gamma_\nu, 1-\gamma_\nu, \frac{1}{\pi|h_\nu|}\right)\quad (h_\nu \neq 0)$$

We now shall deduce an estimate for

$$\sum{}^{*}\left(1-\Phi(f_1(x),\ldots,f_n(x))\right).$$

Using the notation (40), we find from (75) and

$$0 \leqq \varphi_\nu \leqq 1$$

the inequality

$$\Phi = \prod_{\nu=1}^{n}\varphi_\nu = \prod_{\nu=1}^{n}\left(1-(1-\varphi_\nu)\right) \geqq 1 - \sum_{\nu=1}^{n}(1-\varphi_\nu).$$

Hence

$$(80)\quad \sum{}^{*}\left(1-\Phi(f_1,\ldots,f_n)\right) \leqq \sum_{\nu=1}^{n}\sum{}^{*}\left(1-\varphi_\nu(f_\nu)\right).$$

We now fix an arbitrary index σ $(1 \leqq \sigma \leqq n)$ and we denote by K_σ the smallest non-negative integer, which is

$$\geqq \frac{\lambda_\sigma \gamma_\sigma}{2} - 1 .$$

Further we denote by $G_\sigma(k)$ $(k = 0,1,\dots,K_\sigma)$ the parallelepiped which is defined by the inequalities

$$(81) \qquad 0 \leqq z_\nu \leqq \gamma_\nu \quad (\nu = 1,2,\dots\dots,n;\ \nu \neq \sigma),$$

$$(82) \qquad \frac{2k}{\lambda_\sigma} \leqq z_\sigma \leqq \frac{2k+2}{\lambda_\sigma}$$

Then P_0 is entirily covered by the parallelepipeds $G_\sigma(k)$ $(k = 0,1,\dots,K_\sigma)$. In the case that $\frac{2}{\lambda_\sigma} > \gamma_\sigma$, we have $K_\sigma = 0$ and we replace (82) by $0 \leqq z_\sigma \leqq \gamma_\sigma$. Then we shall have

$$(83) \qquad \sum\nolimits^{*} \left(1-\varphi_\sigma(f_\sigma)\right) \leqq \sum_{k=0}^{K_\sigma} N(Q; G_\sigma(k)) \operatorname{Max}\left(1-\varphi_\sigma(z)\right),$$

where the Max is to be extended over $\frac{2k}{\lambda_\sigma} \leqq z \leqq \frac{2k+2}{\lambda_\sigma}$, and

where $N(Q; G_\sigma(k))$ denotes the number of lattice points $(x) \in Q$ for which $(f_1,\dots,f_n)$ (mod 1) falls into $G_\sigma(k)$.

We now fix a value of k $(0 \leqq k \leqq K_\sigma)$ and apply Lemma 4 with $K = 1$ to the parallelepiped $P = G_\sigma(k)$, putting

$$(84) \qquad \alpha_\nu = 0,\ \delta_\nu = \gamma_\nu \quad (\nu = 1,2,\dots\dots,n;\ \nu \neq \sigma),$$

$$(85) \qquad \alpha_\sigma = \frac{2k}{\lambda_\sigma},\ \delta_\sigma = \operatorname{Min}\left(\frac{2}{\lambda_\sigma}, \gamma_\sigma\right).$$

Then $N_p(Q) = N(Q, G_\sigma(k))$ and we find

(86) $N(Q, G_\sigma(k)) \leqq D_\sigma$,

where we have put

(87) $$D_\sigma = 2\prod_{\nu=1}^{n}\left(\eta_\nu^{(\sigma)} + \frac{3}{\lambda_\nu}\right)N(Q) + 2\sum_{(h)}^{*} q_{h_1,1}^{(\sigma)} \cdots\cdots q_{h_n,n}^{(\sigma)} |S(h)|,$$

with

(88) $$\begin{cases} \eta_\nu^{(\sigma)} = \gamma_\nu \quad (\nu = 1,2,\ldots\ldots,n;\ \nu \neq \sigma), \\ \eta_\sigma^{(\sigma)} = \frac{2}{\lambda_\sigma}, \end{cases}$$

(89) $$\begin{cases} q_{0,\nu}^{(\sigma)} = \gamma_\nu + \frac{3}{\lambda_\nu}\ ; \ q_{h_\nu,\nu}^{(\sigma)} = \mathrm{Min}\left(\gamma_\nu + \frac{3}{\lambda_\nu}, 1 - \gamma_\nu + \frac{3}{\lambda_\nu}, \frac{2}{|h_\nu|}\right) \\ \qquad\qquad (h_\nu \neq 0) \quad (\nu = 1,2,\ldots\ldots,n;\ \nu \neq \sigma), \\ q_{0,\sigma}^{(\sigma)} = \frac{5}{\lambda_\sigma},\ q_{h_\sigma,\sigma}^{(\sigma)} = \mathrm{Min}\left(\frac{5}{\lambda_\sigma}, \frac{2}{|h_\sigma|}\right) \quad (h_\sigma \neq 0), \end{cases}$$

whereas $S(h)$ is defined by (46).
Obviously D_σ does not depend on k. Thus for fixed σ the number D_σ is the same for all parallelepipeds $G_\sigma(k)$. We now consider the value of $1 - \varphi_\sigma(z)$ for a number z

of $\frac{2k}{\lambda_\sigma} \leqq z \leqq \frac{2k+2}{\lambda_\sigma}$.

By Lemma 3 we find because of $r = r_\sigma \geqq 4$

$$1-\varphi_\sigma(z)\leqq 1, \text{ if } k=0, \text{ or } k=K_\sigma-1, K_\sigma .$$

$$1-\varphi_\sigma(z)\leqq \frac{1}{128}\left\{\frac{1}{k^{2r-1}}+\frac{1}{(K_\sigma-k)^{2r-1}}\right\}, \text{ if } 1\leqq k\leqq K_\sigma-2 .$$

Therefore we have by (83) and (86)

$$\sum{}^{*}\left(1-\varphi_\sigma(f_\sigma)\right)\leqq D_\sigma\left(3+\frac{1}{64}\sum_{k=1}^{\infty}\frac{1}{k^7}\right)$$

$$\leqq D_\sigma\left(3+\frac{1}{64}\left(1+\int_1^{\infty}\frac{du}{u^7}\right)\right)<\frac{7}{2}D_\sigma .$$

Hence, it follows from (80)

$$\sum{}^{*}\left(1-\Phi(f_1,\ldots\ldots,f_n)\right)\leqq \frac{7}{2}\sum_{\sigma=1}^{n}D_\sigma$$

and therefore from (87)

$$\sum{}^{*}\left(1-\Phi(f_1,\ldots\ldots,f_n)\right)\leqq 7\sum_{\sigma=1}^{n}\prod_{\nu=1}^{n}\left(\eta_\nu^\sigma+\frac{3}{\lambda_\nu}\right)N(Q)$$

$$+7\sum_{(h)}{}^{*}\left\{\sum_{\sigma=1}^{n}q^{(\sigma)}_{h_1,1}\cdots\cdots q^{(\sigma)}_{h_n,n}\right\}|S(h)|$$

and hence a fortiori

$$\leqq\left\{\prod_{\nu=1}^{n}\left(\gamma_\nu+\frac{75}{\lambda_\nu}\right)-\prod_{\nu=1}^{n}\gamma_\nu\right\}N(Q)+$$

$$+\sum_{(h)}{}^{*}\left\{p_{h_1,1}\cdots\cdots p_{h_n,n}-p'_{h_1,1}\cdots\cdots p'_{h_n,n}\right\}|S(h)|$$

because of (88), (89) and Lemma 5. Here the numbers $p_{h_\nu,\nu}$ satisfy (74a) and (74b), whereas the numbers $p'_{h_\nu,\nu}$ are defined by (79). Now we find from (78) immediately

$$N_{P_0}(Q) \leqq \gamma_1 \gamma_2 \cdots\cdots \gamma_n N(Q) + \left\{ \prod_{\nu=1}^{n}\left(\gamma_\nu + \frac{75}{\lambda_\nu}\right) - \prod_{\nu=1}^{n} \gamma_\nu \right\} N(Q)$$

$$+ \sum_{(h)}^{*} p_{h_1,1} \cdots\cdots p_{h_n,n} |S(h)| \,,$$

which proves the lemma in the special case that $P = P_0$. Replacing f_ν by $f_\nu - \alpha_\nu$, we remark that this replacement does not inflict the right hand member of (73), which proves the lemma in full generality.

4. Proof of Theorem 2. Considering the numbers n, $\lambda_1, \ldots, \lambda_n$, $M_1, \ldots, M_n$ which have been defined in Theorem 2, we put

$$\rho_\nu = \mathrm{Min}(n, \lambda_\nu) \qquad (\nu = 1, 2, \ldots\ldots, n),$$

$$r_\nu = \mathrm{Max}(4, [{}^2\log \rho_\nu]) \qquad (\nu = 1, 2, \ldots\ldots, n)$$

and we remark that

$$(90) \qquad \left[\frac{M_\nu}{r_\nu}\right] + 1 = \left[\frac{4\lambda_\nu \log e \rho_\nu}{r_\nu}\right] + 1 > \lambda_\nu \,.$$

We now for $\nu = 1, 2, \ldots, n$ choose an integer $M'_\nu \geqq 1$ by the relation

$$\lambda_\nu = \left[\frac{M'_\nu}{r_\nu}\right] + 1 ,$$

which is always possible, by putting $M'_\nu = r_\nu(\lambda_\nu - 1)$. Then we have by (90)

$$M'_\nu < M_\nu .$$

We now apply the Lemma's 6 and 7, writing there M'_ν instead of M_ν. Then we find the inequality

$$(91)\begin{cases} \left|N^*(Q) - N(Q)\prod_{\nu=1}^{n}(\beta_\nu - \alpha_\nu)\right| \leqq \\ \leqq \left\{\prod_{\nu=1}^{n}\left(\beta_\nu - \alpha_\nu + \frac{75}{\lambda_\nu}\right) - \prod_{\nu=1}^{n}(\beta_\nu - \alpha_\nu)\right\} N(Q) + \\ + \sum_{(h)}^{*} P_{h_1,1} \cdots\cdots P_{h_n,n} |S(h)| , \end{cases}$$

using the notation of the named lemma's. Now because of $M'_\nu < M_\nu$ and (17) the sum $\sum\limits_{(h)}^{*}$ in (91) clearly satisfies the inequality

$$\sum_{(h)}^{*} \leqq T(Q).N(Q)$$

and therefore (20) follows from (91) immediately.

4. Deduction of Theorem 1.

We choose a constant $C > 4$. For each parallelepiped $Q \in \mathcal{S}$, we put

$$2\lambda_\nu = \frac{Cn}{\beta_\nu - \alpha_\nu} , \qquad M'_\nu = 2\lambda_\nu \log 2n .$$

Then we have $M'_\nu > M_\nu$, where M_ν denotes the constant which we have defined in Theorem 2.

Now applying Theorem 2, we find at once

$$\frac{1}{\prod\limits_{\nu=1}^{n}(\beta_\nu-\alpha_\nu)}\left|\frac{N_P(Q)}{N(Q)}-\prod_{\nu=1}^{n}(\beta_\nu-\alpha_\nu)\right| \leq \left\{\prod_{\nu=1}^{n}\left(1+\frac{150}{cn}\right)-1\right\}+$$

$$+\left(1+\frac{150}{cn}\right)^n T(Q,c)$$

$$\leq\left(e^{\frac{150}{c}}-1\right)+e^{\frac{150}{c}}T(Q,c).$$

As c can be chosen arbitrarily large and as $T(Q,c)\to 0$ for a fixed value of c, if Q runs through the sequence S, our assertion (11) follows at once.

NOTES

1) For literature cf my Diophantische Approximationen, Berlin 1936 (In the series Ergebnisse d. Math., IV, 4) Chapter VIII.

2) H. Weyl, Uber die Gleichverteilung von Zahlen mod.Eins.Math.Ann. 77, 313-352 (1916).

3) J.G. van der Corput, Diophantische Ungleichungen, I, Zur Gleichverteilung modulo Eins.Acta math. 56, 373-456 (1931).

4) J.G. van der Corput, Diophantische Ungleichungen, II, Rhythmische Systeme, A und B. Acta math. 59, 209-328 (1932).

5) See my Thesis: Over stelsels Diophantische Ongelijkheden (Diss. Groningen 1930) and for further literature until 1936 cf 1). Further:

J. Teghem, Sur un type d'inégalités diophantiennes. Kon.Ned.Acad.v.Wetensch. Amsterdam, Proc. 42, 147-157 (1939).

J. Teghem, Sommes de Weyl - Sur la Méthode de Vinogradov - van der Corput. Mémoires, Acad. Roy.de Belgique, 23 (1949), fasc. 5.

6) Cf 1), IX Satz 4, p. 101.

7) A. Drewes, Diophantische Benaderingsproblemen. Diss. V.U., Amsterdam 1945.

8) P. Erdös and P. Turán, On a problem in the theory of uniform distribution. Kon.Ned.Akad. v.Wetensch. Amsterdam, Proc. 51, 1146-1154, 1262-1269 (1948) = Indag.Math. 10, 370-378, 406-413 (1948).

9) As usual for real u the symbol $[u]$ in this paper denotes the largest integer $\leq u$.